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A LINEAR PROBLEM OF THE MOTION OF A VORTICITY SOURCE AT THE INTERFACE BETWEEN TWO MEDIA

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The problem of the motion of a vorticity source at the interface between two media is a classical problem of hydrodynamics. The first fundamental results were obtained by N. E. Kochin. In [1], he solved the problem of the motion of a vorticity source under the free surface of a heavy liquid and derived formulas for the buoyancy force, the wave resistance, and the shape of the free surface. In [2], he applied the results of [1] to the case of an interface between two media [2].

In this field of research, some problems are still not clear, in particular, the problem of the motion of a vorticity source above the interface of media, in a lower-density medium. In addition, there are only few data on the calculation of hydrodynamic reactions that affect the vorticity source. This paper is devoted to solution of these problems. For the problem of the motion of a vorticity source under and above the interface of two media, we derive formulas for the complex velocities, the hydrodynamic reactions, and for the shape of the media interface. A highly effective method for calculating the wave integrals is developed. Based on the algorithm proposed for solution of this problem, we performed a numerical experiment to evaluate the influence of the interface of media on the hydrodynamic characteristics of the vorticity source. The asymptotic behavior of the problem solution has also been studied in the far field behind the vorticity source.

1. Let the vorticity source with intensity $C = \Gamma + iQ$ move at a constant velocity near the interface between two media. We assume that the liquid is ideal, incompressible, heavy, and homogeneous in the layers D_1 and D_2. We introduce an inertial coordinate system (related to the vorticity source) by arranging the Ox axis along the undisturbed interface. The problem will be considered in the plane of a complex variable $z = x + iy$. In this case, g is the acceleration of gravity, ρ_k is the liquid density in the kth layer, and $V_{k\infty}$ is the velocity of the liquid at infinity in front of the vorticity source in the layer D_k ($k = 1$ and 2). Let the vorticity source be located at the point $z_0 = x_0 - ih$ of the layer D_1 (motion below the interface of the media. Fig. 1a). For the motion above the interface, the vorticity source is located at the point $z_0 = x_0 + ih$ in the layer D_2 (Fig. 1b).

We introduce the complex velocities $\overline{V}_k(z)$ to describe the disturbed motion of the liquid in the D_k layer ($k = 1$ and 2). The functions $\overline{V}_k(z)$ must be analytical in the D_k region, except for the point z_0 for $k = 1$ (motion under the interface) and $k = 2$ (motion above the interface), and be subject to the following boundary conditions: the continuity of the normal velocity component and of the pressure in going through the interface between the D_1 and D_2 regions and the decay of the disturbed velocities at infinity in front of the vorticity source.

A solution of the problem of the motion under the interface between the media, which was obtained using the general method given in [3], has the form

$$\overline{V}_1(z) = \frac{1}{2\pi i}\frac{C}{z - z_0} + \frac{m_{12}}{2\pi i}\frac{\overline{C}}{z - \overline{z}_0} + \frac{\overline{C}\nu_1 m_{12}^1}{\pi}\int_0^\infty \frac{e^{-i\lambda(z-\overline{z}_0)}}{\lambda - \nu_1}\,d\lambda - \overline{C}\nu_1 m_{12}^1 i e^{-i\nu_1(z-\overline{z}_0)}; \qquad (1.1)$$

$$\overline{V}_2(z) = \frac{V_{2\infty}}{V_{1\infty}}\left\{\frac{m_{12}^1}{\pi i}\frac{C}{z - z_0} - \frac{C\nu_1 m_{12}^1}{\pi}\int_0^\infty \frac{e^{i\lambda(z-z_0)}}{\lambda - \nu_1}\,d\lambda - C\nu_1 m_{12}^1 i e^{i\nu_1(z-z_0)}\right\}, \qquad (1.2)$$

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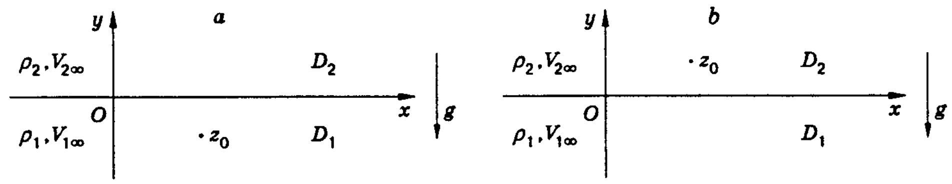

Fig. 1

where

$$m_{12}^1 = \frac{\rho_1 V_{1\infty}^2}{\rho_1 V_{1\infty}^2 + \rho_2 V_{2\infty}^2}; \quad m_{12}^2 = \frac{\rho_2 V_{2\infty}^2}{\rho_1 V_{1\infty}^2 + \rho_2 V_{2\infty}^2}; \quad \nu_1 = \frac{g(\rho_1 - \rho_2)}{\rho_1 V_{1\infty}^2 + \rho_2 V_{2\infty}^2}; \quad m_{12} = m_{12}^1 - m_{12}^2.$$

Using the method of [4], we derive the following formulas for the wave resistance R_x and the buoyancy force R_y of the vorticity source:

$$R_x = -\rho_1 Q V_{1\infty} + \Delta R_x, \quad \Delta R_x = \rho_1 \nu_1 m_{12}^1 (\Gamma^2 + Q^2) e^{-2\nu_1 h}; \tag{1.3}$$

$$R_y = -\rho_1 \Gamma V_{1\infty} + \Delta R_y, \ \Delta R_y = -\frac{\rho_1}{\pi}(\Gamma^2 + Q^2) m_{12}^1 \nu_1 \int_0^\infty \frac{e^{-2\lambda h}}{\lambda - \nu_1} d\lambda - \frac{\rho_1}{4\pi h}(\Gamma^2 + Q^2) m_{12}. \tag{1.4}$$

In this case, ΔR_x and ΔR_y are the supplementary forces for the generalized Joukowski force which acts on the vorticity source.

The shape of the interface is determined by the function

$$f(x) = -\frac{1}{\nu_1 V_{1\infty}} \mathrm{Re}\left\{ m_{12}^1 \overline{V}_1(z) - \frac{V_{1\infty}}{V_{2\infty}} m_{12}^2 \overline{V}_2(z) \right\} \text{ for } z = x. \tag{1.5}$$

Using the general method proposed in [3], we solve similarly the problem of the motion of a vorticity source above the interface:

$$\overline{V}_1(z) = \frac{V_{1\infty}}{V_{2\infty}} \left\{ \frac{m_{12}^2}{\pi i} \frac{C}{z - z_0} - \frac{C \nu_1 m_{12}^2}{\pi} \int_0^\infty \frac{e^{-i\lambda(z - z_0)}}{\lambda - \nu_1} d\lambda - C \nu_1 m_{12}^2 i e^{-i\nu_1 (z - z_0)} \right\}; \tag{1.6}$$

$$\overline{V}_2(z) = \frac{1}{2\pi i} \frac{C}{z - z_0} - \frac{m_{12}}{2\pi i} \frac{\overline{C}}{z - \overline{z}_0} - \frac{\overline{C} \nu_1 m_{12}^2}{\pi} \int_0^\infty \frac{e^{i\lambda(z - \overline{z}_0)}}{\lambda - \nu_1} d\lambda - \overline{C} \nu_1 m_{12}^2 i e^{i\nu_1 (z - \overline{z}_0)}; \tag{1.7}$$

$$R_x = -\rho_2 Q V_{2\infty} + \Delta R_x, \quad \Delta R_x = \rho_2 \nu_1 m_{12}^2 (\Gamma^2 + Q^2) e^{-2\nu_1 h}; \tag{1.8}$$

$$R_y = -\rho_1 \Gamma V_{1\infty} + \Delta R_y, \ \Delta R_y = \frac{\rho_2}{\pi}(\Gamma^2 + Q^2) m_{12}^2 \nu_1 \int_0^\infty \frac{e^{-2\lambda h}}{\lambda - \nu_1} d\lambda - \frac{\rho_2}{4\pi h}(\Gamma^2 + Q^2) m_{12}. \tag{1.9}$$

The interface between two media also obeys formula (1.5).

2. It is interesting to consider the asymptotic behavior of the solution of the problem that we study herein under (1.1) and (1.2) and above (1.6) and (1.7) the interface together with the asymptotics of relation (1.5) for the shape of the interface in the far field behind the vorticity source.

Applying the asymptotic estimates of [5] for the integro-exponential function to the integrals in relations (1.1) and (1.2) with $\mathrm{Re}\, z \gg |z_0|$, we have

$$\overline{V}_1(z) = -2\overline{C} \nu_1 m_{12}^1 i e^{-i\nu_1 (z - \overline{z}_0)} + O\left(\frac{z_0}{z}\right), \quad \overline{V}_2(z) = \frac{V_{2\infty}}{V_{1\infty}} \left\{ -2C \nu_1 m_{12}^1 i e^{i\nu_1 (z - z_0)} \right\} + O\left(\frac{z_0}{z}\right).$$

Substituting these asymptotic relations for the complex velocities into (1.5), we obtain the following asymptotic relation for the interface between the media in the far field:

$$f(x) = 2m_{12}^1 e^{-\nu_1 h} \left\{ \frac{\Gamma}{V_{1\infty}} \sin \nu_1 (x - x_0) + \frac{Q}{V_{1\infty}} \cos \nu_1 (x - x_0) \right\}.$$

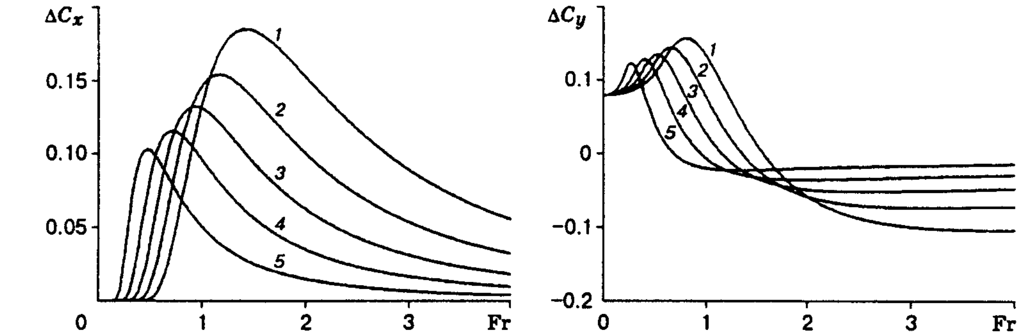

Fig. 2

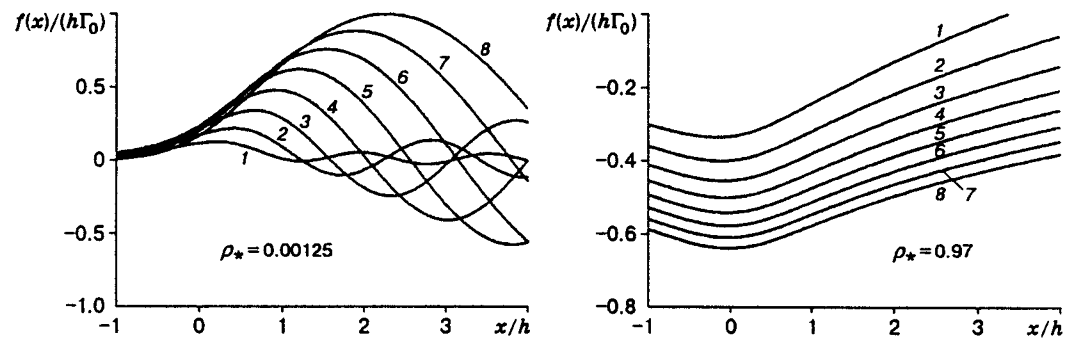

Fig. 3

Clearly, the wave amplitudes for the vortex A_Γ and the source A_Q in the far field obey the relations below

$$A_\Gamma = 2m_{12}^1 \mathrm{e}^{-\nu_1 h}(|\Gamma|/V_{1\infty}), \qquad A_Q = 2m_{12}^1 \mathrm{e}^{-\nu_1 h}(|Q|/V_{1\infty}), \tag{2.1}$$

and the wavelength is $\lambda = 2\pi/\nu_1$.

Note that the resulting asymptotic behavior of the solution of the problem of the motion of a vorticity source under the interface is completely consistent with the results of [2].

Similarly, the asymptotic behavior of expressions (1.5)–(1.7) for the problem of the motion of a vorticity source above the interface is determined:

$$\overline{V}_1(z) = \frac{V_{1\infty}}{V_{2\infty}}\left\{-2C\nu_1 m_{12}^2 i\mathrm{e}^{-i\nu_1(z-z_0)}\right\} + O\left(\frac{z_0}{z}\right), \quad \overline{V}_2(z) = -2\overline{C}\nu_1 m_{12}^2 i\mathrm{e}^{i\nu_1(z-\bar{z}_0)} + O\left(\frac{z_0}{z}\right),$$

$$f(x) = 2m_{12}^2 \mathrm{e}^{-\nu_1 h}\left\{\frac{\Gamma}{V_{2\infty}}\sin\nu_1(x-x_0) + \frac{Q}{V_{2\infty}}\cos\nu_1(x-x_0)\right\}, \tag{2.2}$$

$$A_\Gamma = 2m_{12}^2 \mathrm{e}^{-\nu_1 h}(|\Gamma|/V_{2\infty}), \quad A_Q = 2m_{12}^2 \mathrm{e}^{-\nu_1 h}(|Q|/V_{2\infty}), \quad \lambda = 2\pi/\nu_1.$$

3. To determine the complex velocities (1.1), (1.2) and (1.6), (1.7), it is necessary to calculate integrals of the following form

$$I_1 = \int_0^\infty \frac{\mathrm{e}^{i\lambda z}}{\lambda - \lambda_0}\,d\lambda \ \ (\mathrm{Im}\, z > 0), \qquad I_2 = \int_0^\infty \frac{\mathrm{e}^{-i\lambda z}}{\lambda - \lambda_0}\,d\lambda \ \ (\mathrm{Im}\, z < 0).$$

The main challenge is the fact that there is the oscillating integrand in the above relations. Having substituted the variable $u = -iz(\lambda - \lambda_0)$ $(\mathrm{Im}\, z > 0)$ and $u = iz(\lambda - \lambda_0)$ $(\mathrm{Im}\, z < 0)$, we can change over to the integro-exponential function of a complex argument $I_1 = \mathrm{e}^{i\lambda_0 z}E_1(i\lambda_0 z) - \mathrm{sign}(\mathrm{Re}\, z)\pi i\mathrm{e}^{i\lambda_0 z}$ $(\mathrm{Im}\, z > 0)$, $I_2 = \mathrm{e}^{-i\lambda_0 z}E_1(-i\lambda_0 z) - \mathrm{sign}(\mathrm{Re}\, z)\pi i\mathrm{e}^{-i\lambda_0 z}$ $(\mathrm{Im}\, z < 0)$. The second term on the right-hand side of these expressions appears owing to the singularity of the integrand at the point $\lambda = \lambda_0$.

A highly accurate algorithm was obtained to calculate the integro-exponential function of a complex argument $E_1(\xi)$. We used the Laguerre quadrature formulas [6] for $|\xi| \leqslant 1$ and the method of rational

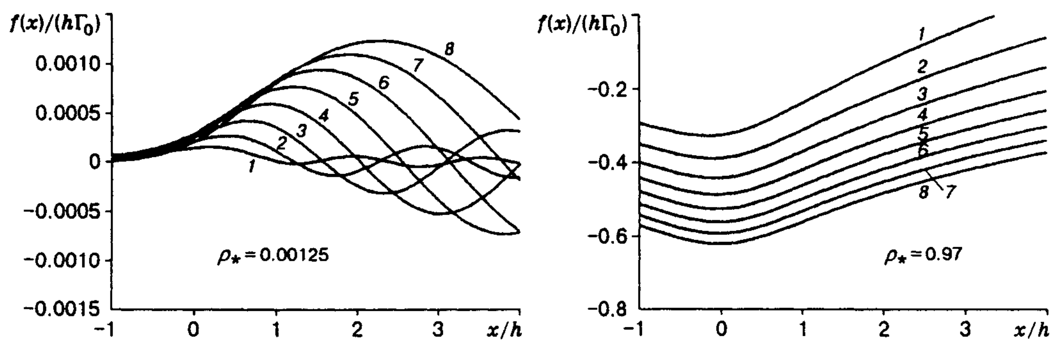

Fig. 4

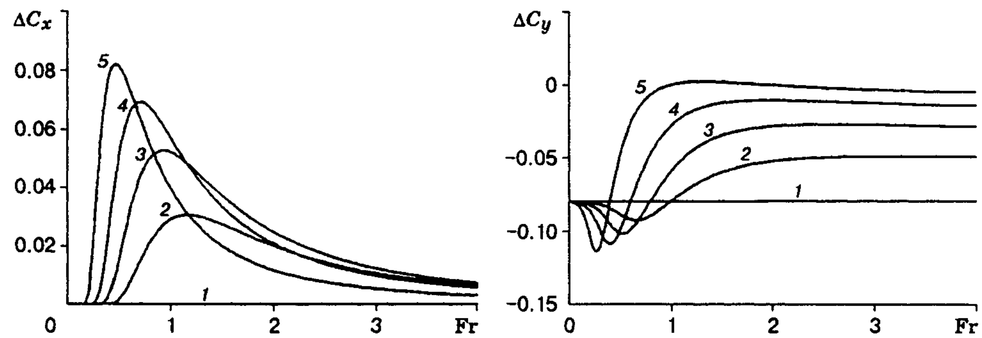

Fig. 5

approximations [7, 8] for $|\xi| > 1$.

4. A highly accurate algorithm of solving the problem was designed on the basis of (1.1)–(1.9), (2.1), and (2.2). For the problem of the motion of a vorticity source under the interface, we have performed comprehensive numerical experiment on the calculation of the coefficients of additional wave resistance $\Delta C_x = \Delta R_x h/(\rho_1(\Gamma^2+Q^2))$, additional buoyancy force $\Delta C_y = \Delta R_y h/(\rho_1(\Gamma^2+Q^2))$, and of the interface shape in the near and far fields behind the vortex. The solution of the problem depends on the following dimensionless parameters: $\mathrm{Fr} = V_{1\infty}/\sqrt{gh}$ is the Froude number, $\rho_* = \rho_2/\rho_1$ is the density ratio, $v_* = V_{2\infty}/V_{1\infty}$ is the ratio of the velocities of incoming flows, and $\Gamma_0 = \Gamma/(V_{1\infty}h)$ is the dimensionless vortex intensity. In calculations, we set $x_0 = 0$.

Figure 2 shows the coefficients ΔC_x and ΔC_y versus the Froude number for $\rho_* = 0$, 0.2, 0.4, 0.6, and 0.8 (curves 1–5) with $v_* = 1$. The following effect was observed: as ρ_* increases, the maxima of the wave resistance and of the buoyancy force decrease and shift toward the smaller Fr values.

In studying the wave problems, it is of interest to calculate the shape of the interface $f(x)/(h\Gamma_0)$. Such calculations were performed for Fr = 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, and 1.2 (curves 1–8 in Fig. 3) with $\rho_* = 0.00125$ and 0.97 (the interfaces between the water and air media and between the saltwater and freshwater, respectively) and $v_* = 1$. The wavelength increases with increasing Froude number and density ratio.

The shape of the interface between two media $f(x)/(h\Gamma_0)$ has also been considered for the problem of the motion of a vorticity source in a less denser liquid with $\mathrm{Fr} = V_{2\infty}/\sqrt{gh} = 0.5$, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, and 1.2 (curves 1–8 in Fig. 4) for $\rho_* = 0.00125$ and 0.97 and $v_* = 1$. One should pay attention to the smaller amplitude of the wave that is generated by the vortex moving above the interface between the water and air media compared with the amplitude of the wave generated by the vortex moving in a denser liquid.

Figure 5 shows curves of the coefficients of additional wave resistance $\Delta C_x = \Delta R_x h/(\rho_2(\Gamma^2+Q^2))$ and of buoyancy force $\Delta C_y = \Delta R_y h/(\rho_2(\Gamma^2+Q^2))$ versus the Froude number for $\rho_* = 0$, 0.2, 0.4, 0.6, and 0.8 (curves 1–5) at $v_* = 1$. The behavior of the coefficient ΔC_x is the same as for the motion under the interface. As for the coefficient ΔC_y, its minimum decreases and shifts toward the smaller Fr values with increasing ρ_*.

Thus, the following conclusion can be drawn: the internal waves which are generated by a vorticity source immersed in a liquid have a substantial effect on the hydrodynamic characteristics of this vorticity source.

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